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Novel Phase Structure of Twisted $O(N)$ ϕ^4 Model on $M^{D-1} \otimes S^1$

Katsuhiko Ohnishi[(a)] [1] and Makoto Sakamoto[(b)] [2]

[(a)] *Graduate School of Science and Technology, Kobe University, Rokkodai, Nada, Kobe 657-8501, Japan*

[(b)] *Department of Physics, Kobe University, Rokkodai, Nada, Kobe 657-8501, Japan*

Abstract

We study the $O(N)$ ϕ^4 model compactified on $M^{D-1} \otimes S^1$, which allows to impose twisted boundary conditions for the S^1-direction. The $O(N)$ symmetry can be broken to H explicitly by the boundary conditions and further broken to I spontaneously by vacuum expectation values of the fields. The symmetries H and I are completely classified and the model turns out to have unexpectedly a rich phase structure. The unbroken symmetry I is shown to depend on not only the boundary conditions but also the radius of S^1, and the symmetry breaking patterns are found to be unconventional. The spontaneous breakdown of the translational invariance is also discussed.

[1] E-mail: `katuhiko@oct.phys.sci.kobe-u.ac.jp`
[2] E-mail: `sakamoto@phys.sci.kobe-u.ac.jp`

1 Introduction

Recently, higher dimensional theories with extra dimensions have revived and have vastly been discussed from various points of view [1, 2, 3]. In many such scenarios, nontrivial backgrounds, such as magnetic flux, vortices, domain walls and branes, turn out to be a key ingredient. It would be of great importance to study physical consequences caused by the nontrivial backgrounds thoroughly.

In this letter, we shall concentrate on a simple situation that "magnetic" flux passes through a circle S^1. Physically, this system may equivalently be described by the system without flux but with fields obeying twisted boundary conditions for the S^1-direction. In the following, we shall take the latter point of view for a technical reason. Even though the situation we consider is very simple, physical consequences caused by boundary conditions turn out to be unexpectedly rich, as we will see later. The parameter space of a model on $M^{D-1} \otimes S^1$ is, in general, wider than that of the model on M^D, and is spanned by twist parameters specifying boundary conditions [4, 5], in addition to parameters appearing in the action. One of characteristic features of such models is the appearance of a critical radius of S^1, at which some of symmetries are broken/restored. The spontaneous breakdown of the translational invariance for the S^1-direction is another characteristic feature [6, 7].

The paper is organized as follows: In the next section, we discuss a general feature of scalar field theories on $M^{D-1} \otimes S^1$ and allowed boundary conditions. In Section 3, the $O(N)$ ϕ^4 model on $M^{D-1} \otimes S^1$ with the antiperiodic boundary condition is studied. In Section 4, general twisted boundary conditions are investigated, and the spontaneous symmetry breaking caused by nonvanishing vacuum expectation values is classified. In Section 5, the model is reanalyzed from a $(D-1)$-dimensional field theory point of view. Section 6 is devoted to conclusions and discussions.

2 A General Discussion

In this section, we shall discuss a general feature of scalar field theories compactified on $M^{D-1} \otimes S^1$. Let us consider an action which consists of N real scalar fields ϕ_i ($i =$

$1, \cdots, N)$[3]

$$S = \int d^{D-1}x \int_0^{2\pi R} dy \left\{ -\frac{1}{2}\partial_A \phi_i(x^\nu, y)\partial^A \phi_i(x^\nu, y) - V(\phi) \right\} \ , \tag{1}$$

where the index A runs from 0 to $D-1$, and x^ν ($\nu = 0, \cdots, D-2$) and y are the coordinates on M^{D-1} and S^1, respectively. The radius of S^1 is denoted by R. Suppose that the action has a symmetry G. Since S^1 is multiply-connected, we can impose a twisted boundary condition on ϕ_i [4, 5] such as

$$\phi_i(x^\nu, y + 2\pi R) = U_{ij}\phi_j(x^\nu, y) \ . \tag{2}$$

The matrix U must belong to G, otherwise the action would not be single-valued. If U is not proportional to the identity matrix, the symmetry group G will be broken to its subgroup H, which consists of all the elements of G commuting with U. Note that this symmetry breaking caused by the boundary condition is not spontaneous but explicit.

In order to find the vacuum configuration of $\phi_i(x^\nu, y)$, one might try to minimize the potential $V(\phi)$. This would, however, lead to wrong vacua in the present model [6, 7]. To find the true vacuum configuration, it is important to take account of the kinetic term in addition to the potential term. This is because the translational invariance could be broken and the vacuum configuration might be coordinate-dependent. Thus, the vacuum configuration will be obtained by solving a minimization problem of the following functional:

$$\mathcal{E}[\phi, R] \equiv \int_0^{2\pi R} dy \left\{ \frac{1}{2}\left(\frac{d\phi_i(y)}{dy}\right)^2 + V(\phi) \right\} \ , \tag{3}$$

where we have assumed that the translational invariance of the uncompactified $(D-1)$-dimensional Minkowski space-time is unbroken.[4]

In general, solving the minimization problem may not be an easy task because we must minimize the functional $\mathcal{E}[\phi, R]$ with the boundary condition (2). Although we have no general procedure to solve the minimization problem, we can present candidates of the vacuum configuration of $\phi_i(y)$ for some class of twisted boundary conditions. Suppose that G is a continuous symmetry and that the matrix U in Eq.(2) can be expressed as $U = e^X$, where X belongs to the algebra of G. (U should continuously be connected to the identity in G.) Then, a candidate of the vacuum configuration will be given by

$$\bar{\phi}_i(y) = (e^{\frac{y}{2\pi R}X})_{ij} v_j \ , \tag{4}$$

[3] Repeated indices are generally summed, unless otherwise indicated.

[4] This is true, at least, at the classical level.

where v_i $(i = 1, \cdots, N)$ are constants. Note that $\bar{\phi}_i(y)$ satisfy the desired boundary condition (2). Even if U cannot continuously be connected to the identity in G, we could find a configuration such as Eq.(4) by restricting some of v_i to zero. In fact, we will see later that the vacuum configuration can be written into the form (4) in the $O(N)$ ϕ^4 model (except for $N = 1$).

3 $O(N)$ ϕ^4 Model with the Antiperiodic Boundary Condition

We shall now investigate the $O(N)$ ϕ^4 model whose potential is given by

$$V(\phi) = -\frac{\mu^2}{2}\phi_i\phi_i + \frac{\lambda}{8}\left(\phi_i\phi_i\right)^2 \ . \tag{5}$$

Since the phase structure is trivial for a positive squared mass, we will assume $\mu^2 > 0$ in the following analysis. The boundary condition for $\phi_i(y)$ is taken to be antiperiodic, i.e.

$$\phi_i(y + 2\pi R) = -\phi_i(y) \qquad \text{for } i = 1, \cdots, N \ . \tag{6}$$

General twisted boundary conditions will be discussed in the next section. Since $U = -\mathbf{1}$, the twisted boundary condition (6) does not break the $O(N)$ symmetry, and hence the unbroken symmetry H, which is consistent with the boundary condition, is $O(N)$ itself.

Let us first consider the case of even N. In this case, it may be convenient to introduce the $N/2$ complex fields by

$$\Phi_a(y) \equiv \frac{e^{-i\frac{y}{2R}}}{\sqrt{2}}\left(\phi_{2a-1}(y) + i\phi_{2a}(y)\right) \qquad \text{for } a = 1, \cdots, \tfrac{N}{2} \ . \tag{7}$$

It should be noticed that $\Phi_a(y)$ obey the periodic boundary condition, i.e.

$$\Phi_a(y + 2\pi R) = +\Phi_a(y) \qquad \text{for } a = 1, \cdots, \tfrac{N}{2} \ . \tag{8}$$

Inserting Eq.(8) into $\mathcal{E}[\phi, R]$, we may write

$$\mathcal{E}[\phi, R] = \mathcal{E}^{(1)}[\Phi, R] + \mathcal{E}^{(2)}[\Phi, R] \ , \tag{9}$$

where

$$\mathcal{E}^{(1)}[\Phi, R] \ \equiv \ \int_0^{2\pi R} dy \left\{ \left|\frac{d\Phi_a}{dy}\right|^2 - \frac{i}{2R}\left(\Phi_a^*\frac{d\Phi_a}{dy} - \frac{d\Phi_a^*}{dy}\Phi_a\right)\right\} \ , \tag{10}$$

$$\mathcal{E}^{(2)}[\Phi, R] \ \equiv \ \int_0^{2\pi R} dy \left\{ \left(\frac{1}{4R^2} - \mu^2\right)|\Phi_a|^2 + \frac{\lambda}{2}\left(|\Phi_a|^2\right)^2 \right\} \ . \tag{11}$$

Our strategy to find the vacuum configuration, which minimizes the functional (9), is as follows: We shall first look for configurations which minimize each of $\mathcal{E}^{(1)}[\Phi, R]$ and $\mathcal{E}^{(2)}[\Phi, R]$, and then construct configurations which minimize both of them simultaneously.

Let us first look for configurations which minimize $\mathcal{E}^{(1)}[\Phi, R]$. To this end, we may expand $\Phi_a(y)$ in the Fourier-series, according to the boundary condition (8), as

$$\Phi_a(y) = \sum_{n\in\mathbf{Z}} \varphi_a^{(n)} e^{i\frac{n}{R}y} \qquad \text{for } a = 1, \cdots, \tfrac{N}{2} \ . \tag{12}$$

Inserting Eq.(12) into $\mathcal{E}^{(1)}[\Phi, R]$, we find

$$\mathcal{E}^{(1)}[\Phi, R] = \frac{2\pi}{R} \sum_{n\in\mathbf{Z}} \left[\left(n + \frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2 \right] |\varphi_a^{(n)}|^2 \ . \tag{13}$$

Since $(n+1/2)^2 - (1/2)^2 \geq 0$ for all $n \in \mathbf{Z}$, $\mathcal{E}^{(1)}[\Phi, R]$ is positive semi-definite. The configuration which gives $\mathcal{E}^{(1)}[\Phi, R] = 0$ is found to be of the form $\Phi_a(y) = \varphi_a^{(0)} + \varphi_a^{(-1)} e^{-i\frac{y}{R}}$ ($a = 1, \cdots, N/2$), where $\varphi_a^{(0)}$ and $\varphi_a^{(-1)}$ are arbitrary complex constants. Let us next look for configurations which minimize $\mathcal{E}^{(2)}[\Phi, R]$. We find that the configuration which minimizes $\mathcal{E}^{(2)}[\Phi, R]$ is $\Phi_a(y) = 0$ for $R \leq 1/(2\mu)$ and $|\Phi_a(y)|^2 = (\mu^2 - 1/(2R)^2)/\lambda$ for $R > 1/(2\mu)$. Combining the above two results and performing an appropriate orthogonal $O(N)$ transformation, we conclude that in terms of ϕ_i the vacuum configuration, which minimizes both of $\mathcal{E}^{(1)}[\Phi, R]$ and $\mathcal{E}^{(2)}[\Phi, R]$ simultaneously, can take to be of the form

$$\langle \phi_i(x^\nu, y) \rangle = \begin{cases} (0, 0, \cdots, 0) & \text{for } R \leq \frac{1}{2\mu} \\ (v\cos(\frac{y}{2R}), v\sin(\frac{y}{2R}), 0, \cdots, 0) & \text{for } R > \frac{1}{2\mu} \end{cases} , \tag{14}$$

where $v = \sqrt{2(\mu^2 - 1/(2R)^2)/\lambda}$. It follows that for $R \leq 1/(2\mu)$ the $O(N)$ symmetry is unbroken, while for $R > 1/(2\mu)$ the spontaneous symmetry breaking occurs and the $O(N)$ symmetry is broken to $O(N-2)$. It is interesting to contrast this result with that of the $O(N)$ ϕ^4 model with the periodic boundary condition, for which the $O(N)$ symmetry is spontaneously broken to $O(N-1)$ irrespective of R.

We now proceed to the case of odd N. In this case, we cannot apply the same method, as was done above, to find the vacuum configuration because we cannot take a complex basis such as Eq.(7) for odd N and because the twist matrix $U = -\mathbf{1}$ cannot continuously be connected to the identity matrix. Nevertheless, we can show that the problem to find the vacuum configuration for odd N reduces to that for even N (expect for $N = 1$). The trick is to add an additional real field $\phi_{N+1}(y)$ satisfying the antiperiodic boundary condition to the action in order to form the $O(N+1)$ ϕ^4 model. It follows from the

previous analysis that the vacuum configuration will be found to be of the form (14) since $N+1$ now becomes an even integer. The fact that the configuration space spanned by $\{\phi_i(y),\, i=1,\cdots,N+1\}$ contains that by $\{\phi_i(y),\, i=1,\cdots,N\}$ implies that the vacuum for odd N is also given by Eq.(14), and hence the spontaneous symmetry breaking from $O(N)$ to $O(N-2)$ can occur for $R>1/(2\mu)$. The exception is the model with $N=1$. In this case, there is no continuous symmetry and the $O(1)$ model has only a discrete symmetry of $G=H=Z_2$. The $O(1)$ ϕ^4 model has been investigated in Ref.[6] and the vacuum configuration has been found to be

$$\langle\phi(x^\nu,y)\rangle = \begin{cases} 0 & \text{for } R \le \frac{1}{2\mu} \\ \frac{2k\mu}{\sqrt{\lambda(1+k^2)}}\, \mathrm{sn}\left(\frac{\mu}{\sqrt{1+k^2}}(y-y_0),k\right) & \text{for } R > \frac{1}{2\mu} \end{cases} . \tag{15}$$

Here, $\mathrm{sn}(u,k)$ is the Jacobi elliptic function whose period is $4K(k)$, where $K(k)$ denotes the complete elliptic function of the first kind. The y_0 is an integration constant and the parameter k $(0\le k<1)$ is determined by the relation $\pi R\mu=\sqrt{1+k^2}K(k)$. Thus, the Z_2 symmetry is unbroken for $R\le 1/(2\mu)$, while it is broken spontaneously for $R>1/(2\mu)$.

4 General Twisted Boundary Conditions

In this section, we shall construct the vacuum configurations of the $O(N)$ ϕ^4 model on $M^{D-1}\otimes S^1$ for general twisted boundary conditions and clarify the phase structure.

To discuss general boundary conditions, it is convenient to transform the matrix U in Eq.(2) by means of a real orthogonal transformation into the normal form. It is known that any matrix U belonging to $O(N)$ can be transformed, by an orthogonal transformation, into a block diagonal form whose diagonal elements are one of 1, -1 and a two dimensional rotation matrix [8]. In this basis, we may arrange the boundary conditions for $\phi_i(y)$ as follows:

$$\begin{aligned}
\phi_a^{(\alpha_0)}(y+2\pi R) &= +\phi_a^{(\alpha_0)}(y) \qquad \text{for } a=1,\cdots,L_0\ , \\
\begin{pmatrix} \phi_{2b_k-1}^{(\alpha_k)}(y+2\pi R) \\ \phi_{2b_k}^{(\alpha_k)}(y+2\pi R) \end{pmatrix} &= \begin{pmatrix} \cos(2\pi\alpha_k) & -\sin(2\pi\alpha_k) \\ \sin(2\pi\alpha_k) & \cos(2\pi\alpha_k) \end{pmatrix} \begin{pmatrix} \phi_{2b_k-1}^{(\alpha_k)}(y) \\ \phi_{2b_k}^{(\alpha_k)}(y) \end{pmatrix} \\
&\qquad \text{for } b_k=1,\cdots,\tfrac{L_k}{2} \text{ and } k=1,\cdots,M-1\ , \\
\phi_c^{(\alpha_M)}(y+2\pi R) &= -\phi_c^{(\alpha_M)}(y) \qquad \text{for } c=1,\cdots,L_M\ ,
\end{aligned} \tag{16}$$

where $L_0+L_1+\cdots+L_{M-1}+L_M=N$ and $0=\alpha_0<\alpha_1<\cdots<\alpha_{M-1}<\alpha_M=1/2$. The

above boundary conditions explicitly break the $O(N)$ symmetry down to

$$H = O(L_0) \times U(\tfrac{L_1}{2}) \times \cdots \times U(\tfrac{L_{M-1}}{2}) \times O(L_M) \tag{17}$$

which is the subgroup of $O(N)$ commuting with the twist matrix U.

Let us first consider the case of $L_0 \neq 0$. In this case, $\phi_a^{(\alpha_0)}(y)$ $(a = 1, \cdots, L_0)$ satisfy the periodic boundary condition. Then, it is easy to show that the vacuum configuration can, without loss of generality, be taken into the form

$$\left\langle \phi_1^{(\alpha_0)}(x^\nu, y) \right\rangle = \sqrt{\frac{2}{\lambda}}\mu \ , \tag{18}$$

and other fields vanish. Thus, the symmetry H in Eq.(17) is spontaneously broken to [5]

$$I = O(L_0 - 1) \times U(\tfrac{L_1}{2}) \times \cdots \times U(\tfrac{L_{M-1}}{2}) \times O(L_M) \ , \tag{19}$$

irrespective of the value of the radius R.

Let us next consider the case of $L_0 = 0$ and $N =$ even. It is then convenient to introduce the $N/2$ complex fields as

$$\Phi_{b_l}^{(\alpha_l)}(y) \equiv \frac{e^{-i\frac{\alpha_l}{R}y}}{\sqrt{2}} \left(\phi_{2b_l-1}^{(\alpha_l)}(y) + i\phi_{2b_l}^{(\alpha_l)}(y) \right) \quad \text{for } b_l = 1, \cdots, \tfrac{L_l}{2} \text{ and } l = 1, \cdots, M \ . \tag{20}$$

Inserting Eqs.(20) into $\mathcal{E}[\phi, R]$, we may rewrite it into the form

$$\mathcal{E}[\phi, R] = \mathcal{E}^{(1)}[\Phi, R] + \mathcal{E}^{(2)}[\Phi, R] + \mathcal{E}^{(3)}[\Phi, R] \ , \tag{21}$$

where

$$\begin{aligned}
\mathcal{E}^{(1)}[\Phi, R] &\equiv \int_0^{2\pi R} dy \left\{ \left| \frac{d\Phi_{b_l}^{(\alpha_l)}}{dy} \right|^2 - i\frac{\alpha_l}{R} \left(\Phi_{b_l}^{(\alpha_l)*} \frac{d\Phi_{b_l}^{(\alpha_l)}}{dy} - \frac{d\Phi_{b_l}^{(\alpha_l)*}}{dy} \Phi_{b_l}^{(\alpha_l)} \right) \right\} \ , \\
\mathcal{E}^{(2)}[\Phi, R] &\equiv \int_0^{2\pi R} dy \left\{ \left[\left(\frac{\alpha_1}{R} \right)^2 - \mu^2 \right] |\Phi_{b_l}^{(\alpha_l)}|^2 + \frac{\lambda}{2} \left(|\Phi_{b_l}^{(\alpha_l)}|^2 \right)^2 \right\} \ , \\
\mathcal{E}^{(3)}[\Phi, R] &\equiv \int_0^{2\pi R} dy \left[\left(\frac{\alpha_l}{R} \right)^2 - \left(\frac{\alpha_1}{R} \right)^2 \right] |\Phi_{b_l}^{(\alpha_l)}|^2 \ .
\end{aligned} \tag{22}$$

Since $(\alpha_1)^2 < (\alpha_l)^2$ for $l = 2, \cdots, M$, it is not difficult to show that in terms of the fields (20) the vacuum configuration which minimizes every $\mathcal{E}^{(j)}[\Phi, R]$ $(j = 1, 2, 3)$ simultaneously can, without loss of generality, be taken into the form

$$\left\langle \Phi_{b_l}^{(\alpha_l)}(x^\nu, y) \right\rangle = \begin{cases} 0 & \text{for } R \leq \frac{\alpha_1}{\mu} \\ \frac{v}{\sqrt{2}} \delta_{\alpha_l, \alpha_1} \delta_{b_l, 1} & \text{for } R > \frac{\alpha_1}{\mu} \end{cases} \tag{23}$$

[5] For $L_0 = 1$, $O(L_0 = 1)$ means Z_2 and the Z_2 symmetry is broken completely.

with $v=\sqrt{2(\mu^2-(\alpha_1/R)^2)/\lambda}$. It follows that for $R \le \alpha_1/\mu$ the symmetry H with $L_0=0$ is unbroken, while for $R>\alpha_1/\mu$ it is spontaneously broken to [6]

$$I = U(\tfrac{L_1}{2}-1)\times U(\tfrac{L_2}{2})\times\cdots\times U(\tfrac{L_{M-1}}{2})\times O(L_M)\ . \tag{24}$$

Let us finally investigate the case of $L_0=0$ and $N=$ odd. To find the vacuum configuration, we may perform the trick used in the previous section: We add an additional real field $\phi_{N+1}(y)$ which satisfies the antiperiodic boundary condition to the action. Then, the resulting model may become the $O(N+1)$ model, which has been analyzed just above since $N+1$ is now even. The result of the $O(N+1)$ model will tell us that the vacuum configuration for the $O(N)$ model with odd N can be taken into the same form as Eq.(23) (except for $N=1$).[7] It follows that for $R\le\alpha_1/\mu$ the symmetry H with $L_0=0$ is unbroken, while for $R>\alpha_1/\mu$ the spontaneous symmetry breaking occurs and the symmetry H is broken to I given in Eq.(24).

5 Reanalysis with Kaluza-Klein Modes

In the previous sections, we have succeeded to reveal the phase structure of the twisted $O(N)$ ϕ^4 model. In this section, we shall reanalyze the model from a $(D-1)$-dimensional field theory point of view, and discuss Nambu-Goldstone modes associated with the broken symmetries and also the symmetry breaking of the translational invariance for the S^1-direction.

To avoid inessential complexities, we shall restrict our considerations to the case of $L_0, L_M=$ even. The N real fields (16) can then form the $N/2$ complex fields which are expanded in the Fourier-series as

$$\frac{1}{\sqrt{2}}\left(\phi^{(\alpha_l)}_{2b_l-1}(x^\nu,y)+i\phi^{(\alpha_l)}_{2b_l}(x^\nu,y)\right)=\sum_{n\in\mathbf{Z}}\varphi^{(\alpha_l)}_{b_l,n}(x^\nu)\ e^{i(\frac{n+\alpha_l}{R})y} \tag{25}$$

for $l=0,1,\cdots,M$ and $b_l=1,2,\cdots,L_l/2$. Inserting Eq.(25) into Eq.(3), we have, up to the quadratic terms with respect to $\varphi^{(\alpha_l)}_{b_l,n}$,

$$\mathcal{E}_0[\varphi,R]=2\pi R\sum_{l=0}^{M}\sum_{b_l=1}^{L_l/2}\sum_{n\in\mathbf{Z}} m^2_{l,n}\ |\varphi^{(\alpha_l)}_{b_l,n}|^2\quad , \tag{26}$$

[6] For $L_1=\cdots=L_{M-1}=0$, the symmetry H is $O(N)$ and is broken to $O(N-2)$ for $R>1/(2\mu)$, as shown in the previous section.

[7] Since for $N=1$ the possible boundary condition is either periodic or antiperiodic, the $O(1)$ model has no new phase more than discussed in the previous section.

where $m_{l,n}^2$ are the squared masses of the Kaluza-Klein modes $\varphi_{b_l,n}^{(\alpha_l)}$ and are given by

$$m_{l,n}^2 = -\mu^2 + \left(\frac{n+\alpha_l}{R}\right)^2 \quad . \tag{27}$$

The second term in Eq.(27) is the Kaluza-Klein mass, which comes from the "kinetic" term $\frac{1}{2}(\partial_y \phi_i(y))^2$ and which gives a positive contribution to the squared mass term.

For $L_0 \neq 0$, the squared mass $m_{0,0}^2$ for the modes $\varphi_{b_0,0}^{(\alpha_0)}$ is always negative irrespective of R. This observation suggests that $\varphi_{b_0,0}^{(\alpha_0)}$ acquire non-vanishing vacuum expectation values, so that the $O(L_0)$ symmetry is spontaneously broken. This is consistent with the results obtained in the previous section. Taking Eq.(18) into account, we should replace the fields $\varphi_{b_l,n}^{(\alpha_l)}$ by $\tilde{\varphi}_{b_l,n}^{(\alpha_l)} + \frac{\mu}{\sqrt{\lambda}}\delta_{l,0}\delta_{b_l,1}\delta_{n,0}$ and then find that all the squared masses for $\tilde{\varphi}_{b_l,n}^{(\alpha_l)}$ become positive semi-definite, as they should be. The $L_0 - 1$ massless modes, $\mathrm{Im}\tilde{\varphi}_{1,0}^{(\alpha_0)}$ and $\tilde{\varphi}_{b_0,0}^{(\alpha_0)}$ $(b_0 = 2, 3, \cdots, L_0/2)$, are found to appear and turn out to correspond to the Nambu-Goldstone modes associated with the broken generators of $O(L_0)/O(L_0 - 1)$.

For $L_0 = 0$, all the squared masses in Eq.(27) are positive for $R < \alpha_1/\mu$. The $m_{1,0}^2$ vanishes at $R = \alpha_1/\mu$ and becomes negative for $R > \alpha_1/\mu$. This is a signal of the phase transition and is consistent with the results obtained in the previous section. Taking Eq.(23) into account, we should replace the fields $\varphi_{b_l,n}^{(\alpha_l)}$ by $\tilde{\varphi}_{b_l,n}^{(\alpha_l)} + \frac{v}{\sqrt{2}}\delta_{l,1}\delta_{b_l,1}\delta_{n,0}$ for $R > \alpha_1/\mu$ and then find that all the squared masses become positive semi-definite, as they should be. The $L_1 - 1$ massless modes, $\mathrm{Im}\tilde{\varphi}_{1,0}^{(\alpha_1)}$ and $\tilde{\varphi}_{b_1,0}^{(\alpha_1)}$ $(b_1 = 2, 3, \cdots, L_1/2)$, are found to appear and turn out to correspond to the Nambu-Goldstone modes associated with the broken generators of $U(\frac{L_1}{2})/U(\frac{L_1}{2} - 1)$. If $L_M = N$, the additional $N-2$ massless modes, $\tilde{\varphi}_{b_M,-1}^{(\alpha_M)}$ $(b_M = 2, 3, \cdots, N/2)$, appear and all the massless modes turn out to form the Nambu-Goldstone modes associated with the broken generators of $O(N)/O(N-2)$.

We shall finally discuss the symmetry breaking of the translational invariance for the S^1-direction. For $L_0 \neq 0$, the vacuum expectation values of the fields are coordinate-independent, so that the translational invariance is unbroken. For $L_0 = 0$, the vacuum expectation values depend on the coordinate y for $R > \alpha_1/\mu$, so that the translational invariance for the S^1-direction is spontaneously broken. It may be instructive to point out that the translational invariance for the S^1-direction can be reinterpreted as a global $U(1)$ symmetry, which is in fact possessed by the theory after compactification. To see this, we note that the translations $y \to y + \epsilon R$ in Eq.(25) can equivalently be realized by the following $U(1)$ transformations:

$$U(1): \quad \varphi_{b_l,n}^{(\alpha_l)} \longrightarrow e^{i(n+\alpha_l)\epsilon}\varphi_{b_l,n}^{(\alpha_l)} \tag{28}$$

from which we may assign a $U(1)$ charge $n+\alpha_l$ to the field $\varphi^{(\alpha_l)}_{b_l,n}$. Thus, the spontaneous breakdown of the translational invariance for the S^1-direction may be understood as that of the $U(1)$ symmetry. For $L_0 \neq 0$, some of $\varphi^{(\alpha_0)}_{b_0,0}$ acquire non-vanishing vacuum expectation values but have no $U(1)$ charges, so that the $U(1)$ symmetry is unbroken. For $L_0 = 0$, some of $\varphi^{(\alpha_1)}_{b_1,0}$ acquire non-vanishing vacuum expectation values for $R > \alpha_1/\mu$. Since $\varphi^{(\alpha_1)}_{b_1,0}$ have the nonzero $U(1)$ charge α_1, the $U(1)$ symmetry would be broken for $R > \alpha_1/\mu$. However, the following modified $U(1)'$ symmetry, which is a combination of the $U(1)$ symmetry and the $O(N)$ symmetry, survives as a symmetry even for $R > \alpha_1/\mu$:

$$U(1)': \quad \varphi^{(\alpha_l)}_{b_l,n} \longrightarrow e^{in\epsilon}\varphi^{(\alpha_l)}_{b_l,n} \quad . \tag{29}$$

This is because $\varphi^{(\alpha_1)}_{b_1,0}$ now have zero $U(1)'$ charge. Hence, no new Nambu-Goldstone modes are produced other than those found before.

6 Conclusions and Discussions

We have studied the $O(N)$ ϕ^4 model compactified on $M^{D-1}\otimes S^1$ with the general twisted boundary conditions. Since S^1 is multiply-connected, the model can be parametrized by not only the mass and the coupling appearing in the action but also the twist matrix appearing in the boundary condition (2). Thus, the parameter space of the $O(N)$ model on $M^{D-1}\otimes S^1$ is much wider than that on M^D. We have succeeded to reveal the rich phase structure and to classify the patterns of the symmetry breaking/restoration thoroughly.

In this letter, our analysis has been restricted to the classical level, and has not taken quantum corrections into account. When the radius R of S^1 is large, R-dependent quantum corrections might be small. But when R is smaller than the inverse of the mass, the leading correction to the squared mass turns out to be proportional to $1/R^2$ for $D = 4$ [9] and hence could drastically change the phase structure at the classical level. Furthermore, the introduction of gauge fields leads to a new interesting feature: Twisted boundary conditions in the directions of the gauge symmetry can dynamically be determined through the Hosotani mechanism [5]. It would be of great interest to analyze R-dependent quantum corrections in gauge field theories and the phase structure of symmetries systematically. The work on these subjects will be reported elsewhere.

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